

On the complexity of failed zero forcing

Yaroslav Shitov

*National Research University Higher School of Economics, 20 Myasnitskaya Ulitsa,
Moscow 101000, Russia*

Abstract

Let G be a simple graph whose vertices are partitioned into two subsets, called 'filled' vertices and 'empty' vertices. A vertex v is said to be forced by a filled vertex u if v is a unique empty neighbor of u . If we can fill all the vertices of G by repeatedly filling the forced ones, then we call an initial set of filled vertices a forcing set. We discuss the so-called failed forcing number of a graph, which is the largest cardinality of a set which is not forcing. Answering the recent question of Ansill, Jacob, Penzellna, Saavedra, we prove that this quantity is NP-hard to compute. Our proof also works for a related graph invariant which is called the skew failed forcing number.

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1. Introduction

This writing is devoted to the problem of zero forcing in graphs, which has recently arisen because of the applications in quantum systems theory [3] and minimum rank problems [1]. In what follows, we denote by G or (V, E) a finite simple graph with vertex set V and edge set E . We assume that the set V is partitioned into two subsets, which we call the *filled* vertices and *empty* vertices.

Let $F \subset V$ be the set of all filled vertices of G . We say that an empty vertex v is *forced by* F if there is a filled vertex u whose unique empty neighbor is v . (A related concept of *skew forcing* is defined analogously but the vertex u is not required to be filled.) A set F is said to be (*skew*) *stalled*

Email address: yaroslav-shitov@yandex.ru (Yaroslav Shitov)

if there is no empty vertex (skew) forced by F . The largest cardinality of a proper (skew) stalled subset of V is called the *(skew) failed forcing number* of G . In [2], Ansill, Jacob, Penzellna, Saavedra posed a problem to determine the computational complexity of these invariants.

2. The result

The goal of this writing is to solve the above mentioned problem. We determine the complexity status of the failed forcing number by proving that the following problems are NP-complete.

Problem 1. (FAILED ZERO FORCING.)

Given: A finite simple graph G and an integer s .

Question: Does G contain a proper stalled subset of cardinality at least s ?

Problem 2. (FAILED SKEW ZERO FORCING.)

Given: A finite simple graph G and an integer s .

Question: Does G contain a proper skew stalled subset of cardinality at least s ?

We prove our result by constructing a polynomial reduction from INDEPENDENT SET to both of the above problems. Recall that a subset $U \subset V$ is called *independent* if the vertices in U are pairwise non-adjacent.

Problem 3. (INDEPENDENT SET.)

Given: A connected simple graph G and an integer c .

Question: Does G contain an independent set of cardinality c ?

We recall that the standard formulation of INDEPENDENT SET does not require G to be connected [4], but the problem remains NP-complete under this restriction. Indeed, we can add to any graph a new vertex adjacent to every other vertex, and this transformation makes the graph connected but does not change the largest cardinality of an independent set.

We proceed with a description of a reduction $G \rightarrow \mathcal{G}$ to Problems 1 and 2. We will assume that G is an instance of INDEPENDENT SET, that is, a connected graph G with the set of vertices V and the set of edges E . We will denote by \mathcal{V}, \mathcal{E} the corresponding sets of \mathcal{G} . We denote by n the cardinality of V , and we set

$$\mathcal{V} = V \cup E^0 \cup \dots \cup E^{2n} \cup \{\varepsilon\},$$

that is, the labels of the vertices of \mathcal{G} are taken from V , from the $2n + 1$ copies of E , and we have one more vertex denoted by ε . We construct the graph \mathcal{G} as follows.

- (1) We subdivide every edge of G . That is, we replace every edge $e = \{u, v\} \in E$ by the two edges $\{u, e^0\}, \{e^0, v\}$.
- (2) For all $e \in E$, we draw a simple path of length $2n + 1$ beginning at e^0 . In other words, we add vertices e^1, \dots, e^{2n} and edges $\{e^i, e^{i+1}\}$ for all i .
- (3) We add the vertex ε and edges $\{\varepsilon, e^0\}$ for all $e \in E$.

We need the following theorem to complete the proof of the main result.

Theorem 4. *If k is the largest cardinality of an independent set of G , then the largest proper stalled subset of \mathcal{G} has cardinality $(2n + 1)|E| + k$. The same conclusion holds for the largest proper skew stalled subset of \mathcal{G} .*

We will give the proof of Theorem 4 in a separate section below. As a corollary of this theorem, we get that

$$(G, k) \rightarrow (\mathcal{G}, (2n + 1)|E| + k)$$

is a polynomial reduction from INDEPENDENT SET to both Problems 1 and 2. In particular, these problems are NP-complete, so the usual and skew failed forcing numbers are NP-hard to compute.

3. The proof

We are going to finalize the paper by proving Theorem 4. Our first lemma establishes the lower bound on the failed forcing number of \mathcal{G} in terms of k . (Here and in the rest of the paper, we denote by k the largest cardinality of an independent set of G .)

Observation 5. *The graph \mathcal{G} contains a skew stalled subset of cardinality $(2n + 1)|E| + k$.*

Proof. Let U be an independent set of G . We need to show that

$$\mathcal{U} = U \cup E^0 \cup \dots \cup E^{2n}$$

is a skew stalled subset of \mathcal{G} . If this is not the case, then some vertex $x \in \mathcal{V} \setminus \mathcal{U}$ is skew forced by \mathcal{U} . This means that either $x = \varepsilon$ or $x \in V \setminus U$, and there is a vertex y for which x is an only neighbor outside \mathcal{U} . We treat the two cases separately.

1. The vertex ε is adjacent only to the vertices e^0 , so $x = \varepsilon$ implies $y = e^0$ with $e = \{a, b\} \in E$. Since x is an only neighbor of y that lives outside \mathcal{U} , the vertices a, b belong to \mathcal{U} . So we have $a, b \in U$ and $\{a, b\} \in E$, which is a contradiction because U is an independent set of G .

2. Now assume $x \in V \setminus U$. The vertices adjacent to x have the form e^0 again, so we get $y = e^0$. This is a contradiction because such a y is adjacent to the vertices $\varepsilon, x \notin \mathcal{U}$. \square

Observation 6. *If a stalled subset \mathcal{S} of \mathcal{G} contains e^i, e^{i+1} , for some e and i , then \mathcal{S} contains e^0, \dots, e^{2n} as well.*

Proof. Assume that the result is not true, which means that $e^t \notin \mathcal{S}$ for some t . If $t < i$, then we choose the maximal $\tau < i$ for which $e^\tau \notin \mathcal{S}$. Then e^τ is the only vertex outside \mathcal{S} which is adjacent to $e^{\tau+1}$. We see that \mathcal{S} forces e^τ , which is impossible because \mathcal{S} is stalled.

Similarly, if $t > i$, then we choose the minimal $\tau > i + 1$ for which $e^\tau \notin \mathcal{S}$. Then e^τ is the only vertex that lives outside \mathcal{S} and is adjacent to $e^{\tau-1}$. We see that \mathcal{S} forces e^τ and get a contradiction. \square

Observation 7. *Let \mathcal{S} be a stalled subset of \mathcal{G} such that $|\mathcal{S}| \geq (2n+1)|E| + 2$. Then \mathcal{S} contains e^0, \dots, e^{2n} for all $e \in E$.*

Proof. Since $|\mathcal{V}| = (2n+1)|E| + n + 1$, there are at most $n - 1$ vertices of \mathcal{G} that lie outside \mathcal{S} . Therefore, there are at least $n + 2$ vertices among e^0, \dots, e^{2n} that belong to \mathcal{S} . In particular, there are two consecutive indexes i, j such that \mathcal{S} contains e^i, e^j . By Observation 6, all the vertices e^0, \dots, e^{2n} belong to \mathcal{S} . \square

Observation 8. *Let \mathcal{S} be a proper stalled subset of \mathcal{G} such that $|\mathcal{S}| \geq (2n+1)|E| + 2$ and $\varepsilon \in \mathcal{S}$. Then $\mathcal{S} \cap V$ is a union of several connected components of G .*

Proof. Assume the converse. Then there are vertices $a, b \in V$ such that $a \in \mathcal{S}$, $b \notin \mathcal{S}$, and $e = \{a, b\} \in E$. By the construction of \mathcal{G} , the vertex e^0 is adjacent to a, b, ε, e^1 . We note that $e^0, e^1 \in \mathcal{S}$ by Observation 7, $\varepsilon \in \mathcal{S}$ by the assumption of the lemma, and $a \in \mathcal{S}$, $b \notin \mathcal{S}$ by the above. We see that b is an only neighbor of e^0 that lies outside \mathcal{S} . So we see that b is forced, which is a contradiction. \square

Observation 9. *Let \mathcal{S} be a proper stalled subset of \mathcal{G} such that $|\mathcal{S}| \geq (2n+1)|E| + 2$. Then $\varepsilon \notin \mathcal{S}$.*

Proof. Assume the converse, which means that $\varepsilon \in \mathcal{S}$. By Observation 7, we have $E^i \subset \mathcal{S}$ for all i . Since the graph G is connected, Observation 8 implies that either $\mathcal{S} \cap V = \emptyset$ or $\mathcal{S} \cap V = V$. The former condition leads to a contradiction with $|\mathcal{S}| \geq (2n+1)|E| + 2$, and the latter is impossible because \mathcal{S} is a proper subset. \square

Observation 10. *Let \mathcal{S} be a proper stalled subset of \mathcal{G} such that $|\mathcal{S}| \geq (2n+1)|E| + 2$. Then $\mathcal{S} \cap V$ is an independent set of G .*

Proof. Assume the converse. We have that $a, b \in \mathcal{S} \cap V$ and $e = \{a, b\} \in E$. By the construction of \mathcal{G} , the vertex e^0 is adjacent to a, b, ε, e^1 . We note that $a, b \in \mathcal{S}$ by the above, $e^0, e^1 \in \mathcal{S}$ by Observation 7, and $\varepsilon \notin \mathcal{S}$ by Observation 9. We see that ε is an only neighbor of e^0 that lies outside \mathcal{S} . So we see that ε is forced, which is a contradiction. \square

Observation 11. *Every proper stalled subset \mathcal{S} of \mathcal{G} has cardinality at most $(2n+1)|E| + k$.*

Proof. If the result was not true, then Observations 9 and 10 would be applicable. We have $\varepsilon \notin \mathcal{S}$ and $|\mathcal{S} \cap V| \leq k$, which means that there are at least $n - k + 1$ vertices outside \mathcal{S} . The total number of vertices of \mathcal{G} is $(2n+1)|E| + n + 1$, so we are done. \square

We note that Observations 5 and 11 complete the proof of Theorem 4. In fact, every skew stalled set is also a stalled set, so the obtained bounds hold for the cardinalities of both usual and skew stalled sets.

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